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Variational Methods for Generating Meshes on Surfaces in Three Dimensions

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Variational methods for generating meshes are developed for arbitrary surfaces in three dimensions. Several variational principles are written down for surfaces, the corresponding elliptic equations are listed, and several simple examples are given. In addition, a short discussion of planar methods and relative scaling of integrals are included in this paper for completeness. © 1986 Academic Press, Inc.

INTRODUCTION

In previous papers variational methods for generating adaptive grids have been discussed. In particular, the paper by Brackbill and Saltzman [1] took the notion of a variational formulation and showed how to use it to generate adaptive meshes in two-dimensional Cartesian geometry. Further work on variational formulations is presented by Saltzman [2]. The latter extended the ideas in [1] by generalizing them to other geometries and higher dimensions. One particular generalization was to formulate the three-dimensional integrals and associated Euler equations. The approach was straightforward and assumed the boundaries and the corresponding point distributions on the boundaries were given. In this paper it will be shown how to generate adaptive meshes on arbitrary surfaces in three dimensions to complement the work that has gone before. It is also the purpose of this paper to show how variational integrals can be simply and elegantly formulated, even though the geometries may be complex and mesh constraints complicated.

The paper will start with a review of two-dimensional Cartesian mesh generation using variational integrals to introduce the critical ideas used in the method. These ideas are then generalized to surfaces in three dimensions in the second section. The third section will show details of the implementation of the method. Careful attention will be paid to scaling, a subject often ignored in mesh generation methods as a whole. The fourth and last section of this paper will exhibit test problems and make some remarks about the problems and the method itself.

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Cartesian Integrals Revisited

We begin with the concept of a mapping from a logical space to a physical space. The idea of mapping from one space to another is not so abstract as the terminology implies. When storing a two-dimensional mesh using FORTRAN, two-dimensional arrays are used. $X(I, J)$ and $Y(I, J)$ would describe a mesh, but also hints of a mapping from a rectangle of integers $\mathrm{IMIN} < I < \mathrm{IMAX}$ and $\mathrm{JMIN} < J < \mathrm{JMAX}$ to some collection of points in (X, Y). By using a simple interpolation scheme the mapping can be "filled in" between the integers to get a smooth mapping. Once the mesh has been "filled in" it is possible to measure various qualities of the mesh using integrals. To differentiate the continuous mappings from the discrete mappings, ξ and η will be used in place of I and J and x and y will be used instead of X and Y. The following integrals measure the smoothness, orthogonality, and volume weighting of a mesh in two-dimensional Cartesian coordinates:

$$I_s = \int (\nabla\xi)^2 + (\nabla\eta)^2 \, dx \, dy \tag{1}$$

$$I_0 = \int (\nabla\xi \cdot \nabla\eta)^2 J^3 \, dx \, dy \tag{2}$$

$$I_v = \int w(x, y) \, J \, dx \, dy \tag{3}$$

where

$$J = \frac{\partial(x, y)}{\partial(\xi, \eta)}.$$

The first integral (Eq. 1) measures the smoothness of the computation mesh by integrating the squares of the gradients of the lines in each coordinate direction. By evenly spacing the lines one naively would conclude that the integral could be minimized. This naive conclusion can be rigorously verified as the minimization of this integral leads to the study of the Dirichlet integral [3]. The second integral (Eq. 2) measures the orthogonality of the coordinate lines. One sees that an orthogonal mesh makes the integral zero, attaining a minimum. The orthogonality measure is blind to a mesh that is folded over on itself. This property implies a non-unique minimum. The third integral (Eq. 3) measures the Jacobian of the mesh against a weight function $w(x, y)$. The naive point of view again is correct in concluding that wherever the weight function is large, the Jacobian and the corresponding line spacing is small. In [1], the nonuniqueness of the minimum of this integral is discussed.

By taking a linear combination of the integral measures, one can control several features of the mapping simultaneously. In the following equation

$$I = \lambda_s I_s + \frac{\lambda_0}{\lambda_0'} I_w + \frac{\lambda_v}{\lambda_v'} I_v \tag{4}$$

the coefficient of the smoothness integral is set to unity while the coefficients of the other two integrals are ratios of two numbers. The numerators of the ratios are chosen to be of order unity while the denominators of the ratios are chosen in a manner to scale the second and third integrals to the size of the first.

A simple yet effective way of calculating the denominators of the coefficients in (4) is to do dimensional analysis. In carrying out a dimensional analysis let l denote length in (x, y) and $\bar{l}$ denote length in (ξ, η). Further let $\bar{w}$ denote some average value of the weight function. By some simple algebra equations (5), (6), and (7) express the dimensions of the given integrands.

$$I_s \sim (\bar{l}/l)^2 \tag{5}$$

$$I_0 \sim (l/\bar{l})^2 \tag{6}$$

$$I_v \sim \bar{w}(l/\bar{l})^2. \tag{7}$$

Finally the denominators λ_0' and λ_v' in Eq. (4) can be easily calculated:

$$\lambda_0' = (l/\bar{l})^4 \qquad \lambda_v' = \bar{w}(l/\bar{l})^4. \tag{8}$$

In practical applications crude integral approximations to the average value of the weight function suffice. For the scale lengths l and $\bar{l}$ the square root of the corresponding areas are adequate. Again a crude quadrature scheme is used to find areas.

With the integrals established, one can turn to mesh generation. To generate a mesh, the linear combination of the integrals are minimized by solving the associated Euler equations. Let

$$F = (\nabla\xi)^2 + (\nabla\eta)^2 + \frac{\lambda_0}{\lambda_0'} (\nabla\xi \cdot \nabla\eta)^2 J^3 + \frac{\lambda_v}{\lambda_v'} wJ \tag{9}$$

be the integrand of the linear combination of integrals. Then

$$\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial x}\frac{\partial}{\partial\xi_x} - \frac{\partial}{\partial y}\frac{\partial}{\partial\xi_y}\right) F = 0 \tag{10}$$

$$\left(\frac{\partial}{\partial\eta} - \frac{\partial}{\partial x}\frac{\partial}{\partial\eta_x} - \frac{\partial}{\partial y}\frac{\partial}{\partial\eta_y}\right) F = 0 \tag{11}$$

is the corresponding set of Euler equations. In practice, the dependent and independent variables of the integrals are interchanged and the Euler equations are written using the interchanged variables. These equations are differenced using centered difference schemes and solved by a relaxation technique. Details of the process can be found in [1]. Although nonuniqueness of solutions minimizing Eqs. (2)–(3) seem an impediment to do numerical work, coupling the integrals with equation (1) works well. It can be hypothesized that the success of the coupling follows from I_s picking the smoothest solution among the nonunique solutions of either I_v or I_0.

Variational Integrals for Surfaces

Instead of generating meshes on planar surfaces, it is sometimes desirable to generate meshes on a given surface. With the framework from the previous section it is easy to generate the proper integrals for general surfaces. First, introduce a parameterization for a surface using the parameters s_1 and s_2:

$$x = x(s_1, s_2) \tag{12}$$

$$y = y(s_1, s_2) \tag{13}$$

$$z = z(s_1, s_2). \tag{14}$$

Suppose that s_1 and s_2 make up an orthogonal parameterization of the given surface and further suppose that s_1 and s_2 are normalized so they each measure arc length along the surface. Along with this parameterization let

$$s_1 = s_1(\xi, \eta) \tag{15}$$

$$s_2 = s_1(\xi, \eta). \tag{16}$$

Here ξ and η again serve the purpose of being the "filled in" array indices as in the previous section. If we had a parameterization like this, the variational integrals could be written down as

$$I_s = \int (\nabla\xi)^2 + (\nabla\eta)^2 \, ds_1 \, ds_2 \tag{17}$$

$$I_0 = \int (\nabla\xi \cdot \nabla\eta)^2 J^3 \, ds_1 \, ds_2 \tag{18}$$

$$I_v = \int wJ \, ds_1 \, ds_2 \tag{19}$$

where

$$J = \frac{\partial(s_1, s_2)}{\partial(\xi, \eta)} \qquad \nabla = \nabla_{s_1, s_2}.$$

The form is exactly the same as the Cartesian case. Unfortunately in practical applications few surfaces are described with orthogonal arc length coordinates. Usually surfaces are interpolated from discrete data. In such cases, the most one can hope for is the following parameterization:

$$x = x(t_1, t_2) \tag{20}$$

$$y = y(t_1, t_2) \tag{21}$$

$$z = z(t_1, t_2). \tag{22}$$

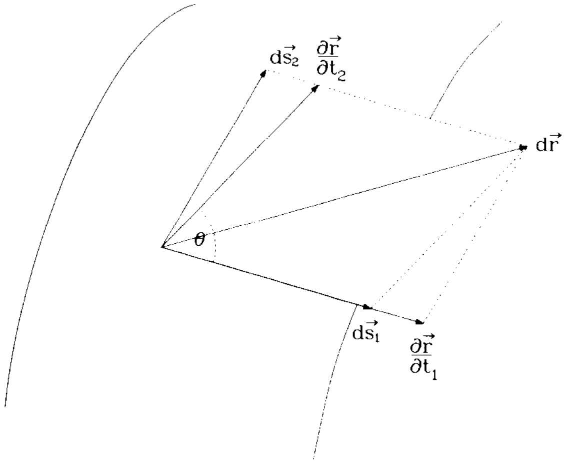

FIG. 1. The geometry of the local transformation creates an orhtogonal projection relative to the tangent $d\mathbf{r}_1$.

Here we assume only that t_1 and t_2 parameterizes the surface in some smooth and nonsingular fashion. We can, however, introduce the followng local transformation:

$$\begin{vmatrix} ds_1 \\ ds_2 \end{vmatrix} = \begin{vmatrix} \left|\frac{\partial \vec{r}}{\partial t_1}\right| & \left|\frac{\partial \vec{r}}{\partial t_2}\right| \cos\theta \\ 0 & \left|\frac{\partial \vec{r}}{\partial t_2}\right| \sin\theta \end{vmatrix} \begin{vmatrix} dt_1 \\ dt_2 \end{vmatrix} \tag{23}$$

where

$$\vec{r} = (x, y, z), \qquad \cos\theta = \frac{\frac{\partial \vec{r}}{\partial t_1} \cdot \frac{\partial \vec{r}}{\partial t_2}}{\left|\frac{\partial \vec{r}}{\partial t_1}\right| \left|\frac{\partial \vec{r}}{\partial t_2}\right|}, \qquad \sin\theta = \frac{\frac{\partial \vec{r}}{\partial t_1} \times \frac{\partial \vec{r}}{\partial t_2}}{\left|\frac{\partial \vec{r}}{\partial t_1}\right| \left|\frac{\partial \vec{r}}{\partial t_2}\right|}.$$

The normalized tangent vector relations are

$$\frac{d\vec{s}_1}{|d\vec{s}_1|} = \frac{d\vec{r}_1}{|d\vec{r}_1|} \qquad \frac{d\vec{s}_2}{|d\vec{s}_2|} = \frac{(d\vec{r}_1 \times d\vec{r}_2) \times d\vec{r}_1}{|(d\vec{r}_1 \times d\vec{r}_2) \times d\vec{r}_1|}. \tag{24}$$

This transformation introduces a local set of orthogonal arc length coordinates. To clarify this transformation, examine Fig. 1.

With these coordinates and using the chain rule the following variational integrals can be derived.

$$I_s=\int\left\{\left[\left(\frac{\partial s_1}{\partial t_1}\frac{\partial t_1}{\partial \xi}+\frac{\partial s_1}{\partial t_2}\frac{\partial t_2}{\partial \xi}\right)^2+\left(\frac{\partial s_1}{\partial t_1}\frac{\partial t_1}{\partial \eta}+\frac{\partial s_1}{\partial t_2}\frac{\partial t_2}{\partial \eta}\right)^2+\left(\frac{\partial s_2}{\partial t_1}\frac{\partial t_1}{\partial \xi}+\frac{\partial s_2}{\partial t_2}\frac{\partial t_2}{\partial \xi}\right)^2\right.\right.$$
$$\left.\left.+\left(\frac{\partial s_2}{\partial t_1}\frac{\partial t_1}{\partial \eta}+\frac{\partial s_2}{\partial t_2}\frac{\partial t_2}{\partial \eta}\right)^2\right]\Big/\left[\frac{\partial(s_1,s_2)}{\partial(t_1,t_2)}\frac{\partial(t_1,t_2)}{\partial(\xi,\eta)}\right]\right\}d\xi\,d\eta \tag{25}$$

$$I_0=\int\left[\left(\frac{\partial s_1}{\partial t_1}\frac{\partial t_1}{\partial \xi}+\frac{\partial s_1}{\partial t_2}\frac{\partial t_2}{\partial \xi}\right)\left(\frac{\partial s_1}{\partial t_1}\frac{\partial t_1}{\partial \eta}+\frac{\partial s_1}{\partial t_2}\frac{\partial t_2}{\partial \eta}\right)\right.$$
$$\left.+\left(\frac{\partial s_2}{\partial t_1}\frac{\partial t_1}{\partial \xi}+\frac{\partial s_2}{\partial t_2}\frac{\partial t_2}{\partial \xi}\right)\left(\frac{\partial s_2}{\partial t_1}\frac{\partial t_1}{\partial \eta}+\frac{\partial s_2}{\partial t_2}\frac{\partial t_2}{\partial \eta}\right)\right]^2 d\xi\,d\eta \tag{26}$$

$$I_v=\int w(t_1,t_2)\left(\frac{\partial(s_1,s_2)}{\partial(t_1,t_2)}\frac{\partial(t_1,t_2)}{\partial(\xi,\eta)}\right)^2 d\xi\,d\eta. \tag{27}$$

It is implicitly assumed that t_1 and t_2 are dependent upon ξ and η.

As in the Cartesian case, in formulating a mesh generator we next scale the integrals. From the form of the integrals it is apparent that the scaling factors are the same as the Cartesian case. That is, if we again use l as scale length on the surface, $\bar{l}$ as the scale length in ξ and η coordinates and $\bar{w}$ as the average value of w, with the linear combination of integrals written as

$$I=\lambda_s I_s+\frac{\lambda_0}{\lambda_0'}I_w+\frac{\lambda_v}{\lambda_v'}I_v \tag{28}$$

the scaling parameters are

$$\lambda_0'=(l/\bar{l})^4 \qquad \lambda_v'=\bar{w}(l/\bar{l})^4. \tag{29}$$

With the combined variational integral written down, the Euler equations can be found using the following differential operators:

$$\left(\frac{\partial}{\partial t_1}-\frac{\partial}{\partial \xi}\frac{\partial}{\partial t_{1_\xi}}-\frac{\partial}{\partial \eta}\frac{\partial}{\partial t_{1_\eta}}\right)F=0 \tag{30}$$

$$\left(\frac{\partial}{\partial t_2}-\frac{\partial}{\partial \xi}\frac{\partial}{\partial t_{2_\xi}}-\frac{\partial}{\partial \eta}\frac{\partial}{\partial t_{2_\eta}}\right)F=0. \tag{31}$$

The differential equations have the form

$$a_{\xi\xi}\frac{\partial^2 t_1}{\partial \xi^2}+a_{\xi\eta}\frac{\partial^2 t_1}{\partial\xi\partial\eta}+a_{\eta\eta}\frac{\partial^2 t_1}{\partial \eta^2}+b_{\xi\xi}\frac{\partial^2 t_2}{\partial \xi^2}+b_{\xi\eta}\frac{\partial^2 t_2}{\partial\xi\partial\eta}+b_{\eta\eta}\frac{\partial^2 t_2}{\partial \eta^2}$$
$$=-d_{t_1t_1}\frac{\partial^2 s_1}{\partial t_1^2}-d_{t_1t_2}\frac{\partial^2 s_1}{\partial t_1\,\partial t_2}-d_{t_2t_2}\frac{\partial^2 s_1}{\partial t_2^2}-e_{t_1t_1}\frac{\partial^2 s_2}{\partial t_1^2}$$
$$-e_{t_1t_2}\frac{\partial^2 s_2}{\partial t_1\,\partial t_2}-e_{t_2t_2}\frac{\partial^2 s_2}{\partial t_2^2}-R_1 \tag{32}$$

$$b_{\xi\xi}\frac{\partial^2 t_1}{\partial \xi^2}+b_{\xi\eta}\frac{\partial^2 t_1}{\partial \xi\,\partial \eta}+b_{\eta\eta}\frac{\partial^2 t_1}{\partial \eta^2}+c_{\xi\xi}\frac{\partial^2 t_2}{\partial \xi^2}+c_{\xi\eta}\frac{\partial^2 t_2}{\partial \xi\,\partial \eta}+c_{\eta\eta}\frac{\partial^2 t_2}{\partial \eta^2}$$
$$= -f_{t_1t_1}\frac{\partial^2 s_1}{\partial t_1^2}-f_{t_1t_2}\frac{\partial^2 s_1}{\partial t_1\,\partial t_2}-f_{t_2t_2}\frac{\partial^2 s_1}{\partial t_2^2}-g_{t_1t_1}\frac{\partial^2 s_2}{\partial t_1^2}$$
$$-g_{t_1t_2}\frac{\partial^2 s_2}{\partial t_1\,\partial t_2}-g_{t_2t_2}\frac{\partial^2 s_2}{\partial t_2^2}-R_2. \tag{33}$$

The coefficients are broken up according to which integral they belong to and are given in the Appendix. Again, since the independent variables are just array indices, the equations can be discretetized using centered differences. The difference equations are then solved using a relaxation scheme.

IMPLEMENTATION

In constructing FORTRAN coding for a surface mesh generator a major consideration of the author was to make use of an older code used for planar meshes. This was rather easy to do since the structure of the equations is the same as the planar case. In fact the only addition is a description of the surface and the calculation of the derivatives of the arc lengths. Of course there are many more first-order terms stemming from the surface geometry. It seems at first that the coefficients of (32)–(33) are very complex. However, many terms are repeated and need be calculated only once. Taking advantage of the great repetition in the terms leads to a great simplification in coding.

The chosen representation of the surface for the test cases given in the following chapter is the tensor product of B-splines. B-splines were chosen for their ease of

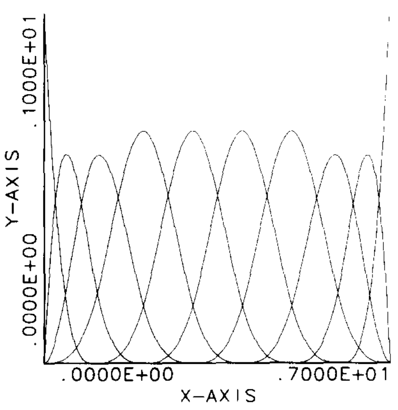

FIG. 2. The set of spline basis functions of order 4 (cubic splines) with knot points at integer points on the interval [0, 7].

use and smoothness. They are adequate for the static examples presented here, but when speed becomes an issue other surface representations should be used. *B*-splines are sums of basis functions which are themselves comprised of piecewise polynomials. In fact they approximate functions the same way that piecewise polynomials do since they are nothing more than a different basis for the piecewise polynomials. Figure 2 displays graphs of the basis functions on a finite interval. Notice that the basis functions are nonzero only over a certain interval. In this example the de Boor spline package was used [4, 5] to generate the graphs with an order of 4 (cubic splines). The spline function as a whole can be written down as

$$f(t) = \sum_i a_i N_i(t) \tag{34}$$

where $N_i(t)$ is the ith basis function and the a_i's are coefficients determined from the approximation problem. For fourth-order splines the function is interpolated at the knot points. Additional conditions are required at the end points of the interval, where we require second derivatives set to zero. A tensor product of the spline functions is then constructed by changing the variable of the spline function and multiplying it with the original spline. The result is in the following equation:

$$f(t_1, t_2) = \sum_{ij} a_{ij} N_i(t_1)\, N_j(t_2). \tag{35}$$

If there are m knots in one direction and n knots in the other then there are $mn + 2(m+n) + 4$ interpolatory conditions on the tensor product. These conditions are met in the following manner. There are mn points to interpolate. As in the one-dimensional case, $2(m+n)$ points can be constrained to have their second derivatives set to zero. The last four constraints are taken care of by setting the cross derivatives at the corners. The following equations summarize these conditions:

$$f(t_{1_k}, t_{2_l}) = \sum_{ij} a_{ij} N_i(t_{1_k})\, N_j(t_{2_l}) \qquad 1 \leqslant k \leqslant m,\ 1 \leqslant l \leqslant n, \tag{36}$$

$$0 = \frac{\partial^2}{\partial t_\alpha^2} f(t_{1_k}, t_{2_l}) = \sum_{ij} a_{ij} \frac{\partial^2}{\partial t_\alpha^2} (N_i(t_{1_k})\, N_j(t_{2_l}))$$

when

$$\alpha = 1, \qquad 1 \leqslant l \leqslant n \text{ when } k = 1, m$$

or

$$\alpha = 2, \qquad 1 \leqslant k \leqslant m \text{ when } l = 1, n, \tag{37}$$

$$0 = \frac{\partial^2}{\partial t_1\, \partial t_2} f(t_{1_k}, t_{2_l}) = \sum_{ij} a_{ij} \frac{\partial^2}{\partial t_1\, \partial t_2} (N_i(t_{1_k})\, N_j(t_{2_l}))$$

for

$$(k, l) = (1 \text{ or } m, 1 \text{ or } n). \tag{38}$$

These equations describe a system of linear equations with a banded structure. The band structure comes from the finite extent of the basis functions. With the above construction, a fairly general surface may be interpolated by using three tensor spline products. The economy in this formulation is that only one *LU* decomposition is done since the coefficients for each coordinate can be found using the same factors. Although the order of the splines are sufficient for calculating all the necessary coefficients in the Euler equations, it is more efficient to difference terms whenever possible. In this implementation, the terms that require first derivatives of the spline function are calculated analytically while higher terms are differenced.

EXAMPLES

Two examples are presented to show that the equations can be succesfully differenced and solved. The first example is a sinusoidal surface with the following parameterization:

$$x = t_1 \tag{39}$$

$$y = t_2 \tag{40}$$

$$z = (t_1 + t_2)/2 + \sin(\pi(t_1 + t_2)) \tag{41}$$

for $0 \leqslant t_1 \leqslant 1$, $0 \leqslant t_2 \leqslant 1$.

With the above equations the spline surface is calculated on some set of knots. The initial mesh is uniform in t_1 and t_2 using integer values. Then the mesh points are perturbed by random numbers whose magnitude is less than a quarter. It is important to remember that even though the points are randomized they still are on the tensor spline surface. Figure 3 displays the randomized surface. Using the difference equations associated only with the smoothness integral, a new mesh is calculated and shown in Fig. 4. As expected, the mesh is uniform.

The second example is a cylinder with the following parameterization:

$$x = (1 + \cos(\pi t_1))/2 \tag{42}$$

$$y = (1 + \sin(\pi t_1))/2 \tag{43}$$

$$z = t_2 \tag{44}$$

for $0 \leqslant t_1 \leqslant 1$, $0 \leqslant t_2 \leqslant 1$.

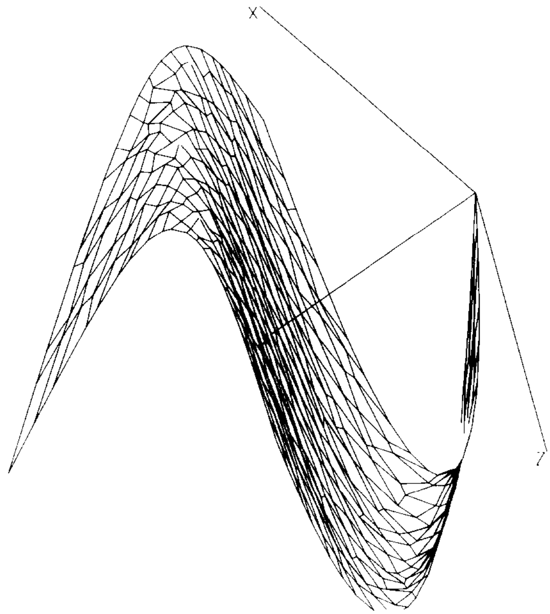

FIG. 3. The surface described by Eqs. (108)–(110) with random perturbations imposed to yield a nonuniform mesh.

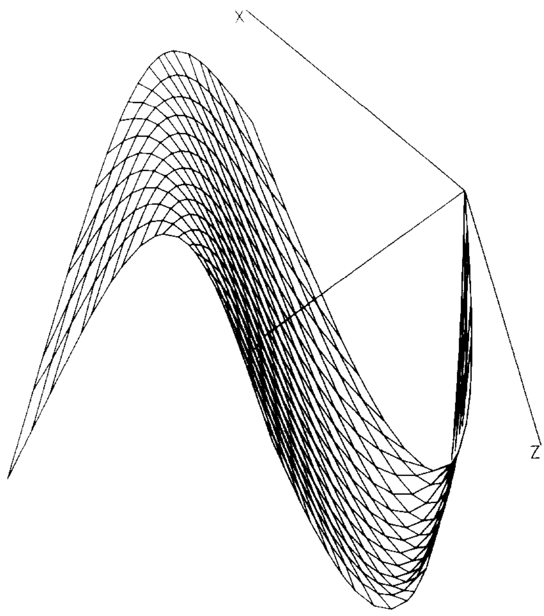

FIG. 4. The surface described by Eqs. (108)–(110) after 20 iterations using just I_s as the variational principle.

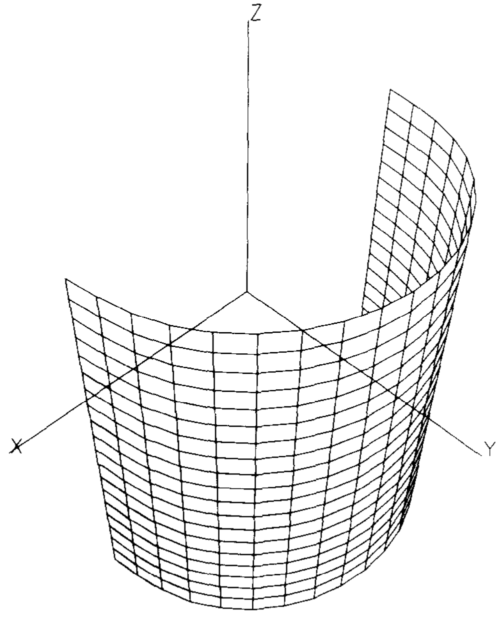

FIG. 5. The surface described by Eqs. (111) (113) is half of a right circular cylinder.

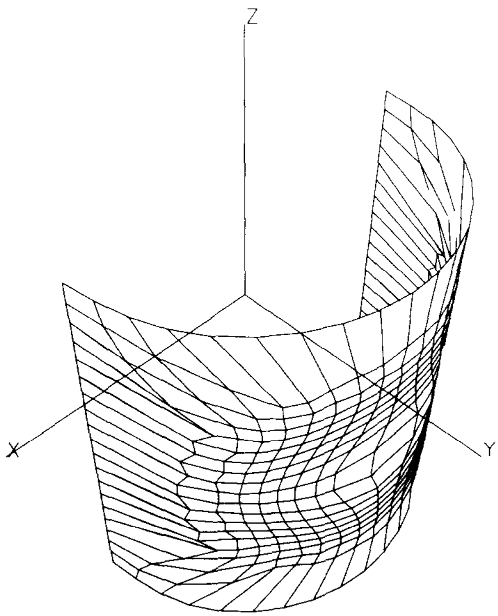

FIG. 6. The surface described by Eqs. (111)–(113) after 20 iterations using a combination of I_s and I_v as the variational principle.

The cylinder is first fitted by three spline functions as in the first example. Next an initial mesh is generated as in Fig. 5. A weight function is now introduced with the following form.

$$w(t_1, t_2) = 50e^{-(14(r-3/14))^2} \qquad r = \sqrt{(t_1 - 1/2)^2 + (t_2 - 1/2)^2}. \tag{45}$$

The effect of this weight function is to bunch the mesh on a circle bent onto the cylinder. The effect of this weight function can be seen in Fig. 6. The whole cylinder seems to be deformed, yet all the points remain on the surface. The illusion is caused by the nonuniform mesh itself. The best way to describe a cylinder is to use a uniform mesh. Instead we insist on placing points around a circular region. The last step in this example is to set the coefficients of the equations associated with the orthogonality integral to a nonzero value. Figure 7 displays the effect of the orthogonality integral on the weight function. Note the greater effect of orthogonality on larger zones than in smaller zones. This is caused by the J^3 term in the integral. In theory one could further increase the value of the orthogonality integral coefficients until the mesh is again uniform, but it has been found that there is an upper limit on both the weighting and orthogonality coefficients. Beyond these limits, the solution of the Euler equations by relaxation does not converge. These limits also depend on the number of mesh points in the problem. As the number of mesh points increase, the limits become larger.

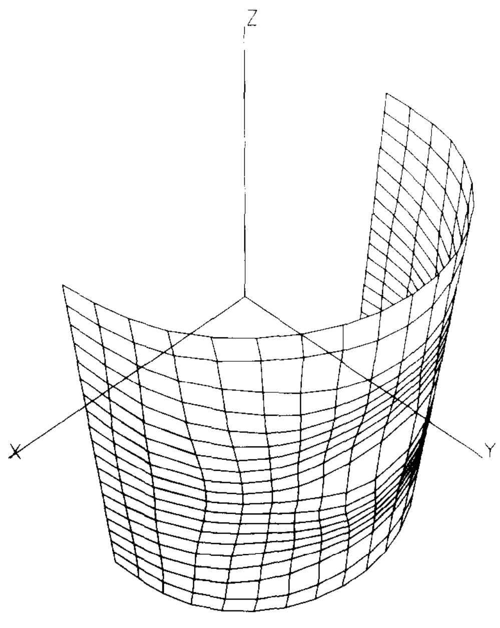

FIG. 7. The surface described by Eqs. (111)–(113) after 20 iterations using a combination of I_s, I_v, and I_0 as the variational principle. Mesh lines can be made more orthogonal without degrading the effect of I_v.

Finally, a few comments should be made about the performance of the current scheme. As stated earlier, the major problem with the current scheme is the representation of the surface. The *B*-spline representation is slow compared with other parts of the coding. The spline representation is adequate for generating meshes that are afterwards fixed, but is probably too slow for changing the mesh every cycle of a hydrodynamics problem. Other ways to represent the surface should be found for time dependent problems.

CONCLUSION

The results show there are ways to generate meshes on a large variety of surfaces using variational principles. It is also the hope of the author that the reader is convinced that it is conceptually simpler to construct mesh generation schemes by using variational integrals rather than by constructing the elliptic equations directly.

APPENDIX

The coefficients of Eqs. (32)–(33) are given below and are grouped with the corresponding variational integral. Letting

$$J_s = \frac{\partial s_1}{\partial t_1}\frac{\partial s_2}{\partial t_2} - \frac{\partial s_1}{\partial t_2}\frac{\partial s_2}{\partial t_1} \quad J_t = \frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta} - \frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi} \quad J = J_s J_t$$

the terms from I_s are

$$a_{\xi\xi} = -2J_s\left(\frac{\partial t_2^2}{\partial \xi} + \frac{\partial t_2^2}{\partial \eta}\right)\left(\frac{\partial s_1^2}{\partial \eta} + \frac{\partial s_2^2}{\partial \eta}\right)\Big/(J^2 J_t) \tag{A1}$$

$$a_{\xi\eta} = 4J_s\left(\frac{\partial t_2^2}{\partial \xi} + \frac{\partial t_2^2}{\partial \eta}\right)\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta} + \frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\Big/(J^2 J_t) \tag{A2}$$

$$a_{\eta\eta} = -2J_s\left(\frac{\partial t_2^2}{\partial \xi} + \frac{\partial t_2^2}{\partial \eta}\right)\left(\frac{\partial s_1^2}{\partial \xi} + \frac{\partial s_2^2}{\partial \xi}\right)\Big/(J^2 J_t) \tag{A3}$$

$$b_{\xi\xi} = 2J_s\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \xi} + \frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \eta}\right)\left(\frac{\partial s_2^2}{\partial \eta} + \frac{\partial s_2^2}{\partial \eta}\right)\Big/(J^2 J_t) \tag{A4}$$

$$b_{\xi\eta} = -4J_s\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \xi} + \frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \eta}\right)\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta} + \frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\Big/(J^2 J_t) \tag{A5}$$

$$b_{\eta\eta} = 2J_s\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \xi} + \frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \eta}\right)\left(\frac{\partial s_1^2}{\partial \xi} + \frac{\partial s_2^2}{\partial \xi}\right)\Big/(J^2 J_t) \tag{A6}$$

$$c_{\xi\xi} = -2J_s\left(\frac{\partial t_1}{\partial \xi}^2 + \frac{\partial t_1}{\partial \eta}^2\right)\left(\frac{\partial s_2}{\partial \eta}^2 + \frac{\partial s_2}{\partial \eta}^2\right)\Big/(J^2 J_t) \tag{A7}$$

$$c_{\xi\eta} = 4J_s\left(\frac{\partial t_1}{\partial \xi}^2 + \frac{\partial t_1}{\partial \eta}^2\right)\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta} + \frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\Big/(J^2 J_t) \tag{A8}$$

$$c_{\eta\eta} = -2J_s\left(\frac{\partial t_1}{\partial \xi}^2 + \frac{\partial t_1}{\partial \eta}^2\right)\left(\frac{\partial s_1}{\partial \xi}^2 + \frac{\partial s_2}{\partial \xi}^2\right)\Big/(J^2 J_t) \tag{A9}$$

$$d_{t_1 t_1} = -2J_t\left(\frac{\partial s_1}{\partial t_2}^2 + \frac{\partial s_2}{\partial t_2}^2\right)\left(\frac{\partial s_2}{\partial \xi}\frac{\partial t_2}{\partial \xi} + \frac{\partial s_2}{\partial \eta}\frac{\partial t_2}{\partial \eta}\right)\Big/J^2 \tag{A10}$$

$$d_{t_1 t_2} = 4J_t\left(\frac{\partial s_1}{\partial t_1}\frac{\partial s_1}{\partial t_2} + \frac{\partial s_2}{\partial t_1}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_2}{\partial \xi}\frac{\partial t_2}{\partial \xi} + \frac{\partial s_2}{\partial \eta}\frac{\partial t_2}{\partial \eta}\right)\Big/J^2 \tag{A11}$$

$$d_{t_2 t_2} = -2J_t\left(\frac{\partial s_1}{\partial t_1}^2 + \frac{\partial s_2}{\partial t_1}^2\right)\left(\frac{\partial s_2}{\partial \xi}\frac{\partial t_2}{\partial \xi} + \frac{\partial s_2}{\partial \eta}\frac{\partial t_2}{\partial \eta}\right)\Big/J^2 \tag{A12}$$

$$e_{t_1 t_1} = 2J_t\left(\frac{\partial s_1}{\partial t_2}^2 + \frac{\partial s_2}{\partial t_2}^2\right)\left(\frac{\partial s_1}{\partial \xi}\frac{\partial t_2}{\partial \xi} + \frac{\partial s_1}{\partial \eta}\frac{\partial t_2}{\partial \eta}\right)\Big/J^2 \tag{A13}$$

$$e_{t_1 t_2} = -4J_t\left(\frac{\partial s_1}{\partial t_1}\frac{\partial s_1}{\partial t_2} + \frac{\partial s_2}{\partial t_1}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_1}{\partial \xi}\frac{\partial t_2}{\partial \xi} + \frac{\partial s_1}{\partial \eta}\frac{\partial t_2}{\partial \eta}\right)\Big/J^2 \tag{A14}$$

$$e_{t_2 t_2} = 2J_t\left(\frac{\partial s_1}{\partial t_1}^2 + \frac{\partial s_2}{\partial t_1}^2\right)\left(\frac{\partial s_1}{\partial \xi}\frac{\partial t_2}{\partial \xi} + \frac{\partial s_1}{\partial \eta}\frac{\partial t_2}{\partial \eta}\right)\Big/J^2 \tag{A15}$$

$$f_{t_1 t_1} = 2J_t\left(\frac{\partial s_1}{\partial t_2}^2 + \frac{\partial s_2}{\partial t_2}^2\right)\left(\frac{\partial s_2}{\partial \xi}\frac{\partial t_1}{\partial \xi} + \frac{\partial s_2}{\partial \eta}\frac{\partial t_1}{\partial \eta}\right)\Big/J^2 \tag{A16}$$

$$f_{t_1 t_2} = -4J_t\left(\frac{\partial s_1}{\partial t_1}\frac{\partial s_1}{\partial t_2} + \frac{\partial s_2}{\partial t_1}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_2}{\partial \xi}\frac{\partial t_1}{\partial \xi} + \frac{\partial s_2}{\partial \eta}\frac{\partial t_1}{\partial \eta}\right)\Big/J^2 \tag{A17}$$

$$f_{t_2 t_2} = 2J_t\left(\frac{\partial s_1}{\partial t_1}^2 + \frac{\partial s_2}{\partial t_1}^2\right)\left(\frac{\partial s_2}{\partial \xi}\frac{\partial t_1}{\partial \xi} + \frac{\partial s_2}{\partial \eta}\frac{\partial t_1}{\partial \eta}\right)\Big/J^2 \tag{A18}$$

$$g_{t_1 t_1} = -2J_t\left(\frac{\partial s_1}{\partial t_2}^2 + \frac{\partial s_2}{\partial t_2}^2\right)\left(\frac{\partial s_1}{\partial \xi}\frac{\partial t_1}{\partial \xi} + \frac{\partial s_1}{\partial \eta}\frac{\partial t_1}{\partial \eta}\right)\Big/J^2 \tag{A19}$$

$$g_{t_1 t_2} = 4J_t\left(\frac{\partial s_1}{\partial t_1}\frac{\partial s_1}{\partial t_2} + \frac{\partial s_2}{\partial t_1}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_1}{\partial \xi}\frac{\partial t_1}{\partial \xi} + \frac{\partial s_1}{\partial \eta}\frac{\partial t_1}{\partial \eta}\right)\Big/J^2 \tag{A20}$$

$$g_{t_2 t_2} = -2J_t\left(\frac{\partial s_1}{\partial t_2}^2 + \frac{\partial s_2}{\partial t_2}^2\right)\left(\frac{\partial s_1}{\partial \xi}\frac{\partial t_1}{\partial \xi} + \frac{\partial s_1}{\partial \eta}\frac{\partial t_1}{\partial \eta}\right)\Big/J^2 \tag{A21}$$

$$R_1 = 0 \tag{A22}$$

$$R_2 = 0. \tag{A23}$$

The terms from I_v are

$$a_{\xi\xi} = -2wJ_s^2 \frac{\partial t_2}{\partial \eta}^2 \tag{A24}$$

$$a_{\xi\eta} = 4wJ_s^2 \frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta} \tag{A25}$$

$$a_{\eta\eta} = -2wJ_s^2 \frac{\partial t_2}{\partial \xi}^2 \tag{A26}$$

$$b_{\xi\xi} = 2wJ_s^2 \frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \eta} \tag{A27}$$

$$b_{\xi\eta} = -4wJ_s^2 \left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta} + \frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\right)\Big/ 2 \tag{A28}$$

$$b_{\eta\eta} = 2wJ_s^2 \frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \xi} \tag{A29}$$

$$c_{\xi\xi} = -2wJ_s^2 \frac{\partial t_1}{\partial \eta}^2 \tag{A30}$$

$$c_{\xi\eta} = 4wJ_s^2 \frac{\partial t_1}{\partial \xi}\frac{\partial t_1}{\partial \eta} \tag{A31}$$

$$c_{\eta\eta} = -2wJ_s^2 \frac{\partial t_1}{\partial \xi}^2 \tag{A32}$$

$$d_{t_1 t_1} = -2\frac{\partial s_2}{\partial t_2} wJ_s J_t^2 \tag{A33}$$

$$d_{t_1 t_2} = 2\frac{\partial s_2}{\partial t_1} wJ_s J_t^2 \tag{A34}$$

$$d_{t_2 t_2} = 0 \tag{A35}$$

$$e_{t_1 t_1} = 2\frac{\partial s_1}{\partial t_2} wJ_s J_t^2 \tag{A36}$$

$$e_{t_1 t_2} = -2\frac{\partial s_1}{\partial t_1} wJ_s J_t^2 \tag{A37}$$

$$e_{t_2 t_2} = 0 \tag{A38}$$

$$f_{t_1 t_1} = 0 \tag{A39}$$

$$f_{t_1 t_2} = -2\frac{\partial s_2}{\partial t_2} wJ_s J_t^2 \tag{A40}$$

$$f_{t_2 t_2} = 2 \frac{\partial s_2}{\partial t_1} w J_s J_t^2 \tag{A41}$$

$$g_{t_1 t_1} = 0 \tag{A42}$$

$$g_{t_1 t_2} = 2 \frac{\partial s_1}{\partial t_2} w J_s J_t^2 \tag{A43}$$

$$g_{t_2 t_2} = -2 \frac{\partial s_1}{\partial t_1} w J_s J_t^2 \tag{A44}$$

$$R_1 = -J^2 \frac{\partial w}{\partial t_1} \tag{A45}$$

$$R_2 = -J^2 \frac{\partial w}{\partial t_2}. \tag{A46}$$

The terms from I_0 are

$$a_{\xi\xi} = -2 \left(\frac{\partial s_1}{\partial t_1} \frac{\partial s_1}{\partial \eta} + \frac{\partial s_2}{\partial t_1} \frac{\partial s_2}{\partial \eta} \right)^2 \tag{A47}$$

$$\begin{aligned} a_{\xi\eta} = & -4 \left(\frac{\partial s_1}{\partial t_1} \frac{\partial s_1}{\partial \eta} + \frac{\partial s_2}{\partial t_1} \frac{\partial s_2}{\partial \eta} \right) \left(\frac{\partial s_1}{\partial t_1} \frac{\partial s_1}{\partial \xi} + \frac{\partial s_2}{\partial t_1} \frac{\partial s_2}{\partial \xi} \right) \\ & -4 \left(\frac{\partial s_1}{\partial t_1}^2 + \frac{\partial s_2}{\partial t_1}^2 \right) \left(\frac{\partial s_1}{\partial \xi} \frac{\partial s_1}{\partial \eta} + \frac{\partial s_2}{\partial \xi} \frac{\partial s_2}{\partial \eta} \right) \end{aligned} \tag{A48}$$

$$a_{\eta\eta} = -2 \left(\frac{\partial s_1}{\partial t_1} \frac{\partial s_1}{\partial \xi} + \frac{\partial s_2}{\partial t_1} \frac{\partial s_2}{\partial \xi} \right)^2 \tag{A49}$$

$$b_{\xi\xi} = -2 \left(\frac{\partial s_1}{\partial t_1} \frac{\partial s_1}{\partial \eta} + \frac{\partial s_2}{\partial t_1} \frac{\partial s_2}{\partial \eta} \right) \left(\frac{\partial s_1}{\partial t_2} \frac{\partial s_1}{\partial \eta} + \frac{\partial s_2}{\partial t_2} \frac{\partial s_2}{\partial \eta} \right) \tag{A50}$$

$$\begin{aligned} b_{\xi\eta} = & -4 \left(\frac{\partial s_1}{\partial \xi} \frac{\partial s_1}{\partial \eta} + \frac{\partial s_2}{\partial \xi} \frac{\partial s_2}{\partial \eta} \right) \left(\frac{\partial s_1}{\partial t_1} \frac{\partial s_1}{\partial t_2} + \frac{\partial s_2}{\partial t_1} \frac{\partial s_2}{\partial t_2} \right) \\ & -2 \left(\frac{\partial s_1}{\partial t_1} \frac{\partial s_1}{\partial \eta} + \frac{\partial s_2}{\partial t_1} \frac{\partial s_2}{\partial \eta} \right) \left(\frac{\partial s_1}{\partial t_2} \frac{\partial s_1}{\partial \xi} + \frac{\partial s_2}{\partial t_2} \frac{\partial s_2}{\partial \xi} \right) \\ & -2 \left(\frac{\partial s_1}{\partial t_1} \frac{\partial s_1}{\partial \xi} + \frac{\partial s_2}{\partial t_1} \frac{\partial s_2}{\partial \xi} \right) \left(\frac{\partial s_1}{\partial t_2} \frac{\partial s_1}{\partial \eta} + \frac{\partial s_2}{\partial t_2} \frac{\partial s_2}{\partial \eta} \right) \end{aligned} \tag{A51}$$

$$b_{\eta\eta} = -2 \left(\frac{\partial s_1}{\partial t_1} \frac{\partial s_1}{\partial \xi} + \frac{\partial s_2}{\partial t_1} \frac{\partial s_2}{\partial \xi} \right) \left(\frac{\partial s_1}{\partial t_2} \frac{\partial s_1}{\partial \xi} + \frac{\partial s_2}{\partial t_2} \frac{\partial s_2}{\partial \xi} \right) \tag{A52}$$

$$c_{\xi\xi} = -2 \left(\frac{\partial s_1}{\partial t_2} \frac{\partial s_1}{\partial \eta} + \frac{\partial s_2}{\partial t_2} \frac{\partial s_2}{\partial \eta} \right)^2 \tag{A53}$$

$$c_{\xi\eta} = -4\left(\frac{\partial s_1}{\partial t_2}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial t_2}\frac{\partial s_2}{\partial \eta}\right)\left(\frac{\partial s_1}{\partial t_2}\frac{\partial s_1}{\partial \xi}+\frac{\partial s_2}{\partial t_2}\frac{\partial s_2}{\partial \xi}\right) -4\left(\frac{\partial s_2{}^1}{\partial t_2}+\frac{\partial s_2{}^2}{\partial t_2}\right)\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right) \tag{A54}$$

$$c_{\eta\eta} = -2\left(\frac{\partial s_1}{\partial t_2}\frac{\partial s_1}{\partial \xi}+\frac{\partial s_2}{\partial t_2}\frac{\partial s_2}{\partial \xi}\right)^2 \tag{A55}$$

$$d_{t_1t_1} = -2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\left(2\frac{\partial s_1}{\partial t_1}\frac{\partial t_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}+\frac{\partial s_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}+\frac{\partial s_1}{\partial \eta}\frac{\partial t_1}{\partial \xi}\right) -2\left(\frac{\partial s_1}{\partial \eta}\frac{\partial s_1}{\partial t_1}+\frac{\partial s_2}{\partial \eta}\frac{\partial s_2}{\partial t_1}\right)\left(\frac{\partial s_1}{\partial \eta}\frac{\partial t_1{}^2}{\partial \xi}+\frac{\partial s_1}{\partial \xi}\frac{\partial t_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}\right) -2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial t_1}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial t_1}\right)\left(\frac{\partial s_2}{\partial \eta}\frac{\partial t_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}+\frac{\partial s_1}{\partial \xi}\frac{\partial t_1{}^2}{\partial \eta}\right) \tag{A56}$$

$$d_{t_1t_2} = -2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\left(2\frac{\partial s_1}{\partial t_1}\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\right)+\frac{\partial s_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}+\frac{\partial s_1}{\partial \xi}\frac{\partial t_2}{\partial \eta}\right) -2\left(\frac{\partial s_1}{\partial \eta}\frac{\partial s_1}{\partial t_1}+\frac{\partial s_2}{\partial \eta}\frac{\partial s_2}{\partial t_1}\right)\left(2\frac{\partial s_1}{\partial \eta}\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \xi}+\frac{\partial s_1}{\partial \xi}\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\right)\right) -2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial t_1}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial t_1}\right)\left(\frac{\partial s_1}{\partial \eta}\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\right)+2\frac{\partial s_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \eta}\right) \tag{A57}$$

$$d_{t_2t_2} = -2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\left(2\frac{\partial s_1}{\partial t_1}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}\right) -2\left(\frac{\partial s_1}{\partial \eta}\frac{\partial s_1}{\partial t_1}+\frac{\partial s_2}{\partial \eta}\frac{\partial s_2}{\partial t_1}\right)\left(\frac{\partial s_1}{\partial \eta}\frac{\partial t_2{}^2}{\partial \xi}+\frac{\partial s_1}{\partial \xi}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}\right) -2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial t_1}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial t_1}\right)\left(\frac{\partial s_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial s_1}{\partial \xi}\frac{\partial t_2{}^2}{\partial \eta}\right) \tag{A58}$$

$$e_{t_1t_1} = -2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\left(2\frac{\partial s_2}{\partial t_1}\frac{\partial t_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial t_1}{\partial \eta}+\frac{\partial s_2}{\partial \eta}\frac{\partial t_1}{\partial \xi}\right) -2\left(\frac{\partial s_1}{\partial \eta}\frac{\partial s_1}{\partial t_1}+\frac{\partial s_2}{\partial \eta}\frac{\partial s_2}{\partial t_1}\right)\left(\frac{\partial s_2}{\partial \eta}\frac{\partial t_1{}^2}{\partial \xi}+\frac{\partial s_2}{\partial \xi}\frac{\partial t_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}\right) -2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial t_1}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial t_1}\right)\left(\frac{\partial s_2}{\partial \eta}\frac{\partial t_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial t_1{}^2}{\partial \eta}\right) \tag{A59}$$

$$e_{t_1t_2}= -2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\left(2\frac{\partial s_2}{\partial t_1}\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\right)+\frac{\partial s_2}{\partial \eta}\frac{\partial t_2}{\partial \xi}+\frac{\partial s_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}\right)$$
$$-2\left(\frac{\partial s_1}{\partial \eta}\frac{\partial s_1}{\partial t_1}+\frac{\partial s_2}{\partial \eta}\frac{\partial s_2}{\partial t_1}\right)\left(2\frac{\partial s_2}{\partial \eta}\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \xi}+\frac{\partial s_2}{\partial \xi}\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\right)\right)$$
$$-2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial t_1}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial t_1}\right)\left(\frac{\partial s_2}{\partial \eta}\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\right)+2\frac{\partial s_2}{\partial \xi}\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \eta}\right) \quad \text{(A60)}$$

$$e_{t_2t_2}= -2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\left(2\frac{\partial s_2}{\partial t_1}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}\right)$$
$$-2\left(\frac{\partial s_1}{\partial \eta}\frac{\partial s_1}{\partial t_1}+\frac{\partial s_2}{\partial \eta}\frac{\partial s_2}{\partial t_1}\right)\left(\frac{\partial s_2}{\partial \eta}\frac{\partial t_2{}^2}{\partial \xi}+\frac{\partial s_2}{\partial \xi}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}\right)$$
$$-2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial t_1}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial t_1}\right)\left(\frac{\partial s_2}{\partial \eta}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial t_2{}^2}{\partial \eta}\right) \quad \text{(A61)}$$

$$f_{t_1t_1}= -2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\left(2\frac{\partial s_1}{\partial t_2}\frac{\partial t_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}\right)$$
$$-2\left(\frac{\partial s_1}{\partial \eta}\frac{\partial s_1}{\partial t_2}+\frac{\partial s_2}{\partial \eta}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_1}{\partial \eta}\frac{\partial t_1{}^2}{\partial \xi}+\frac{\partial s_1}{\partial \xi}\frac{\partial t_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}\right)$$
$$-2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial t_2}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_1}{\partial \eta}\frac{\partial t_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}+\frac{\partial s_1}{\partial \xi}\frac{\partial t_1{}^2}{\partial \eta}\right) \quad \text{(A62)}$$

$$f_{t_1t_2}= -2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\left(2\frac{\partial s_1}{\partial t_2}\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\right)+\frac{\partial s_1}{\partial \eta}\frac{\partial t_1}{\partial \xi}+\frac{\partial s_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}\right)$$
$$-2\left(\frac{\partial s_1}{\partial \eta}\frac{\partial s_1}{\partial t_2}+\frac{\partial s_2}{\partial \eta}\frac{\partial s_2}{\partial t_2}\right)\left(2\frac{\partial s_1}{\partial \eta}\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \xi}+\frac{\partial s_1}{\partial \xi}\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\right)\right)$$
$$-2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial t_2}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_1}{\partial \eta}\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\right)+2\frac{\partial s_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \eta}\right) \quad \text{(A63)}$$

$$f_{t_2t_2}= -2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\left(2\frac{\partial s_1}{\partial t_2}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial s_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}+\frac{\partial s_1}{\partial \xi}\frac{\partial t_1}{\partial \eta}\right)$$
$$-2\left(\frac{\partial s_1}{\partial \eta}\frac{\partial s_1}{\partial t_2}+\frac{\partial s_2}{\partial \eta}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_1}{\partial \eta}\frac{\partial t_2{}^2}{\partial \xi}+\frac{\partial s_1}{\partial \xi}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}\right)$$
$$-2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial t_2}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial s_1}{\partial \xi}\frac{\partial t_2{}^2}{\partial \eta}\right) \quad \text{(A64)}$$

$$g_{t_1t_1}=-2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\left(2\frac{\partial s_2}{\partial t_2}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial s_2}{\partial \eta}\frac{\partial t_2}{\partial \xi}+\frac{\partial s_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}\right)$$
$$-2\left(\frac{\partial s_1}{\partial \eta}\frac{\partial s_1}{\partial t_2}+\frac{\partial s_2}{\partial \eta}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_2}{\partial \eta}\frac{\partial t_2}{\partial \xi}^2+\frac{\partial s_2}{\partial \xi}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}\right)$$
$$-2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial t_2}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_2}{\partial \eta}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}^2\right) \tag{A65}$$

$$g_{t_1t_2}=-2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\left(2\frac{\partial s_2}{\partial t_2}\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\right)+\frac{\partial s_2}{\partial \eta}\frac{\partial t_2}{\partial \xi}+\frac{\partial s_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}\right)$$
$$-2\left(\frac{\partial s_1}{\partial \eta}\frac{\partial s_1}{\partial t_2}+\frac{\partial s_2}{\partial \eta}\frac{\partial s_2}{\partial t_2}\right)\left(2\frac{\partial s_2}{\partial \eta}\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \xi}+\frac{\partial s_2}{\partial \xi}\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\right)\right)$$
$$-2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial t_2}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_2}{\partial \eta}\left(\frac{\partial t_1}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \xi}\right)+2\frac{\partial s_2}{\partial \xi}\frac{\partial t_1}{\partial \eta}\frac{\partial t_2}{\partial \eta}\right) \tag{A66}$$

$$g_{t_2t_2}=-2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial \eta}\right)\left(2\frac{\partial s_2}{\partial t_2}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial s_2}{\partial \eta}\frac{\partial t_2}{\partial \xi}+\frac{\partial s_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}\right)$$
$$-2\left(\frac{\partial s_1}{\partial \eta}\frac{\partial s_1}{\partial t_2}+\frac{\partial s_2}{\partial \eta}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_2}{\partial \eta}\frac{\partial t_2}{\partial \xi}^2+\frac{\partial s_2}{\partial \xi}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}\right)$$
$$-2\left(\frac{\partial s_1}{\partial \xi}\frac{\partial s_1}{\partial t_2}+\frac{\partial s_2}{\partial \xi}\frac{\partial s_2}{\partial t_2}\right)\left(\frac{\partial s_2}{\partial \eta}\frac{\partial t_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}+\frac{\partial s_2}{\partial \xi}\frac{\partial t_2}{\partial \eta}^2\right) \tag{A67}$$

$$R_1=0 \tag{A68}$$

$$R_2=0. \tag{A69}$$

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